

Online Minimum Spanning Trees with Weight Predictions

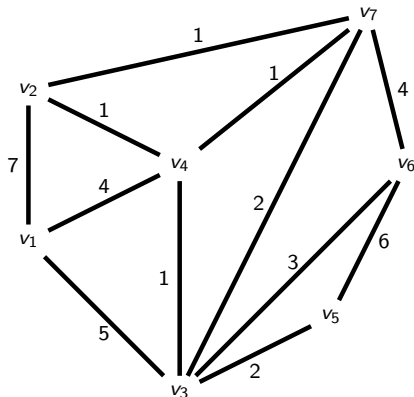
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University of Southern Denmark

WADS 2023

The Minimum Spanning Tree Problem

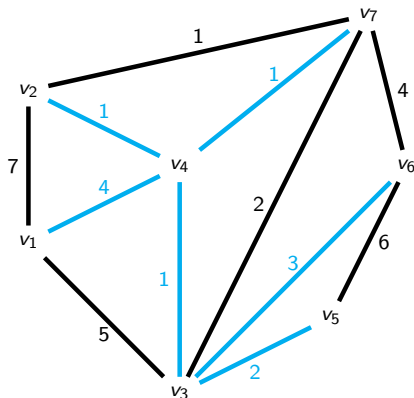
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Objective: Construct spanning tree of minimum cost.



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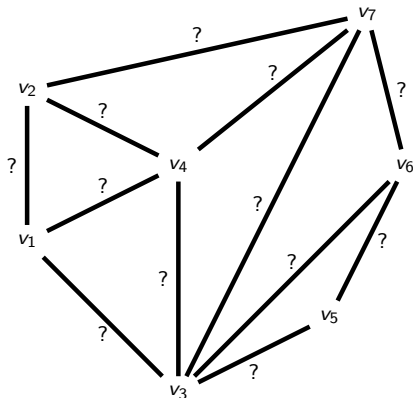


In this case, we have that $\text{OPT}(G) = 12$.

Online MST Problem - Weight Arrival Model (WMST)

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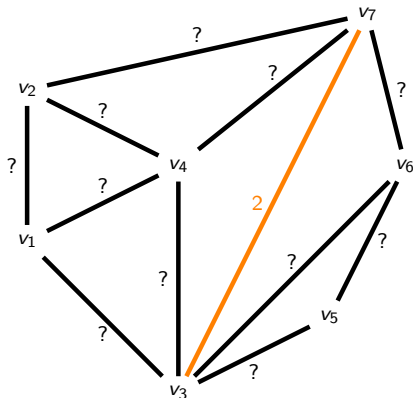
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- Sequence of $(w(e), e)$



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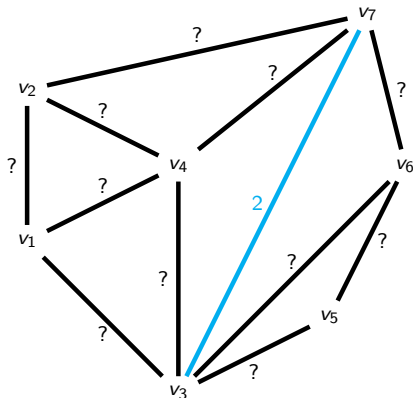
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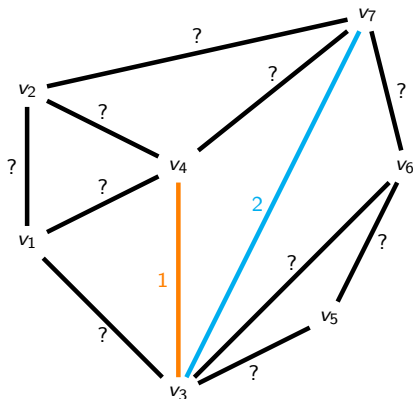
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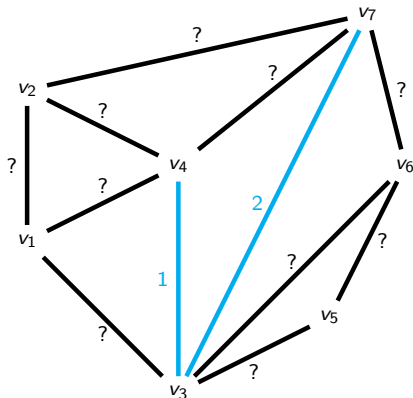
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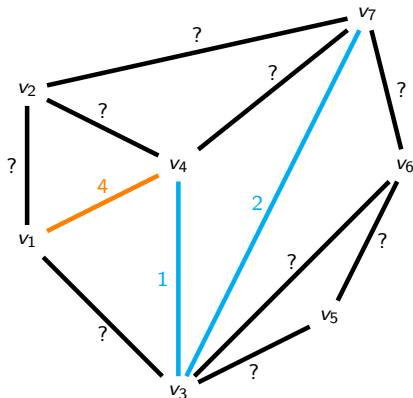
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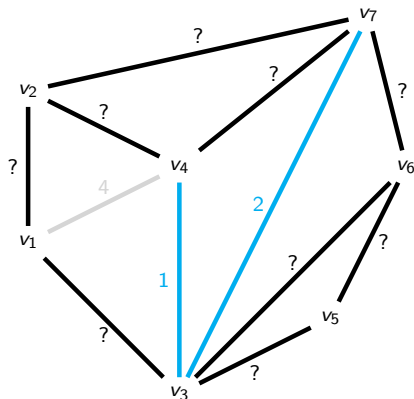
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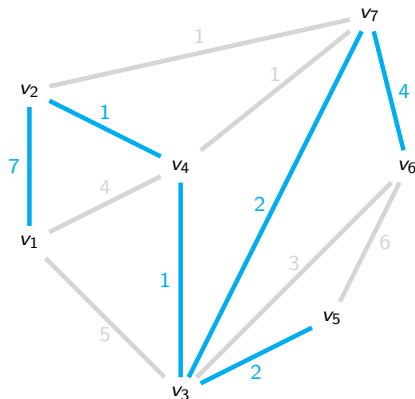
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Online MST Problem - Weight Arrival Model (WMST)

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In this case $\text{ALG}(G) = 17$, whereas $\text{OPT}(G) = 12$.

Comparing Online Algorithms

Given an online algorithm, ALG , for an online minimization problem, we typically measure the quality of ALG by its *competitive ratio*:

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Definition

An online algorithm, ALG , for a minimization problem Π is said to be *c-competitive* if there exists a constant b such that for all instances I of Π :

$$\text{ALG}(I) \leq c \cdot \text{OPT}(I) + b.$$

The *competitive ratio* of ALG is then

$$CR_{\text{ALG}} = \inf\{c \mid \text{ALG is } c\text{-competitive}\}.$$

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For the WMST problem, no online algorithm is competitive.

Online Algorithms with Predictions

- Competitive analysis: Optimize for worst case.
- Machine Learning: Optimize for common cases.
- Question is: can we combine the best of both worlds?

In recent years, a lot of work has been done on *Online Algorithms with Predictions*.

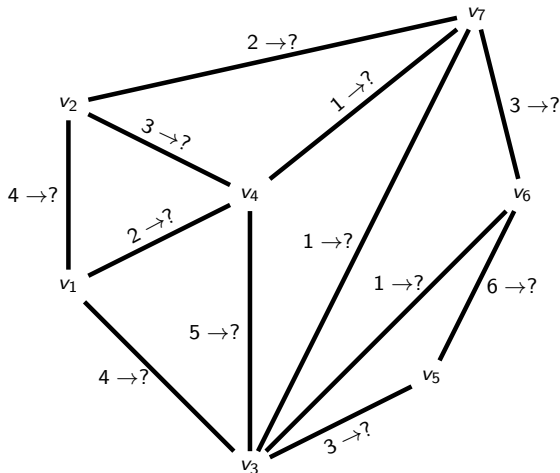
The idea is to assume that the online algorithms can access some predictions providing (unreliable) information about the instance.

Formally, these predictions need not be Machine Learned.

Our playground

An instance:

$G = (V, E, \hat{w})$, where $\hat{w}: E \rightarrow \mathbb{R}^+$ is a prediction for the edge weights.
Sequence of $(w(e), e)$.



Details of this framework

- Error η : **sum of the $n - 1$ greatest prediction errors**, where $n = |V(G)|$.
- $\varepsilon = \frac{\eta}{\text{OPT}}$.

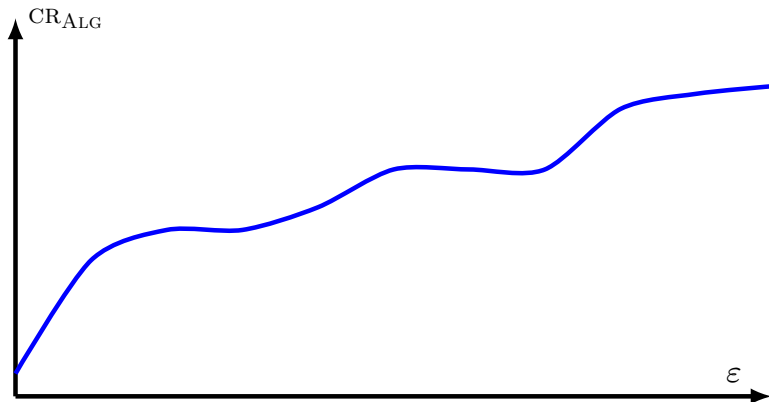
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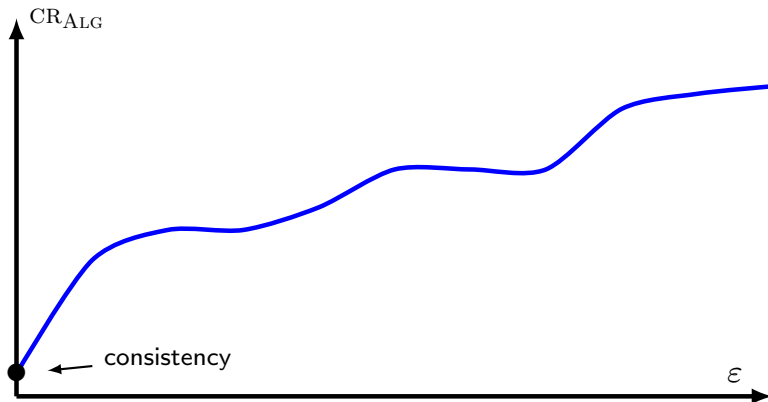
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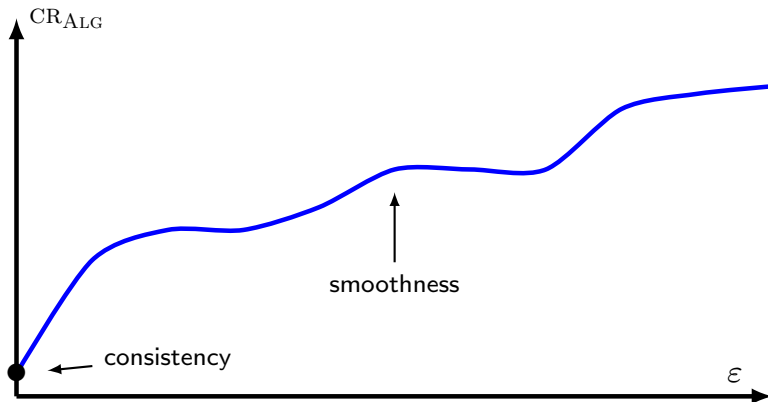
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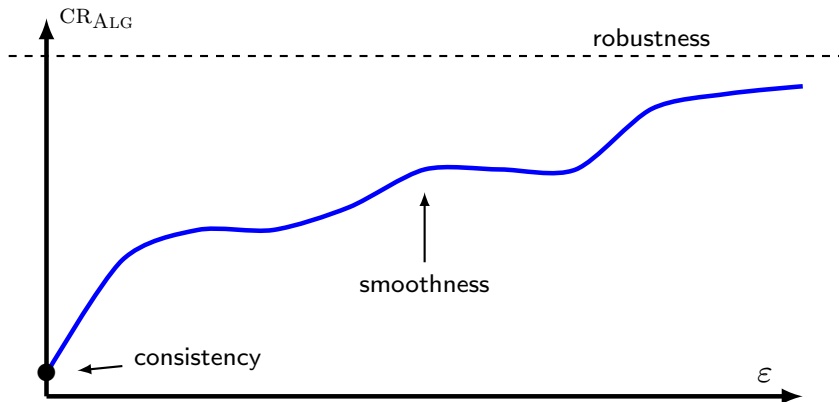
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Follow-the-predictions

Algorithm 1 FTP

- 1: **Input:** A WMST-instance (G, \hat{w})
- 2: Let \hat{T} be a MST of G w.r.t. \hat{w}
- 3: **while** receiving inputs $(w(e_i), e_i)$ **do**
- 4: **if** $e_i \in \hat{T}$ **then**
- 5: Accept e_i

▷ Add e_i to the solution

Greedy-FTP

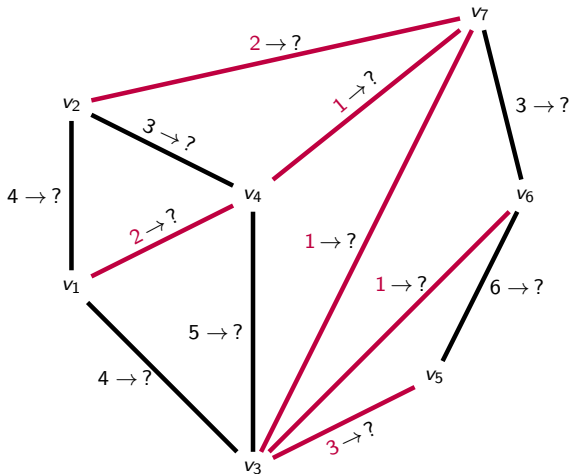
Algorithm 2 GFTP

```
1: Input: A WMST-instance  $(G, \hat{w})$ 
2: Let  $T_G$  be a MST of  $G$  w.r.t.  $\hat{w}$ 
3:  $U = E(G)$  ▷  $U$  contains the unseen edges
4: while receiving inputs  $(w(e_i), e_i)$  do
5:    $U = U \setminus \{e_i\}$ 
6:   if  $e_i \in T_G$  then ▷ Add  $e_i$  to the solution
7:     Accept  $e_i$ 
8:   else if  $e_i \notin T_G$  then
9:      $C$  is the cycle  $e_i$  introduces in  $T_G$ 
10:     $C_U = U \cap C$ 
11:    if  $C_U \neq \emptyset$  then
12:       $e_{\max} = \arg \max_{e_j \in C_U} \{\hat{w}(e_j)\}$ 
13:      if  $w(e_i) \leq \hat{w}(e_{\max})$  then ▷ Update  $T_G$ 
14:         $T_G = (T_G \setminus \{e_{\max}\}) \cup \{e_i\}$  ▷ Add  $e_i$  to the solution
15:        Accept  $e_i$ 
```

Running GFTP on our example graph.

Color codes:

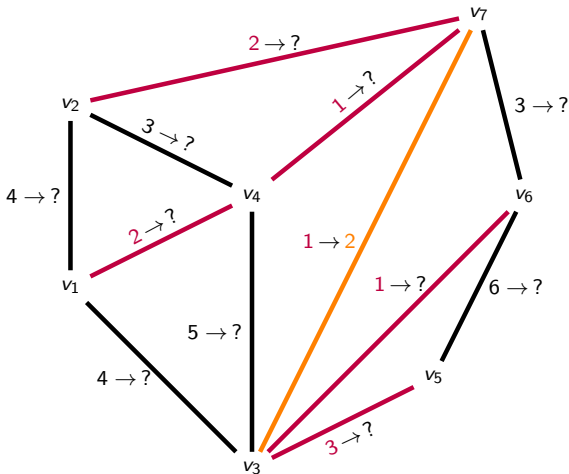
- *T*
- Just revealed
- Accepted by GFTP
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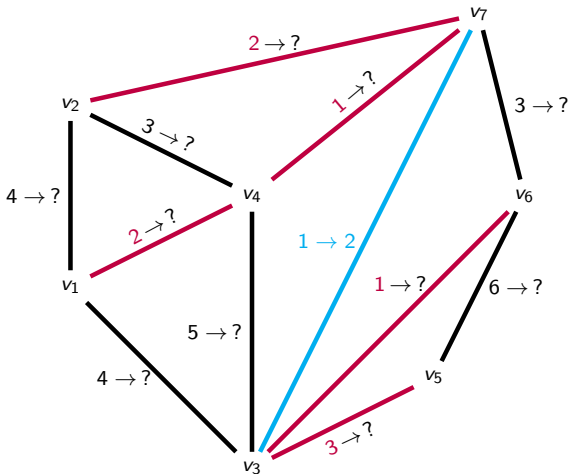
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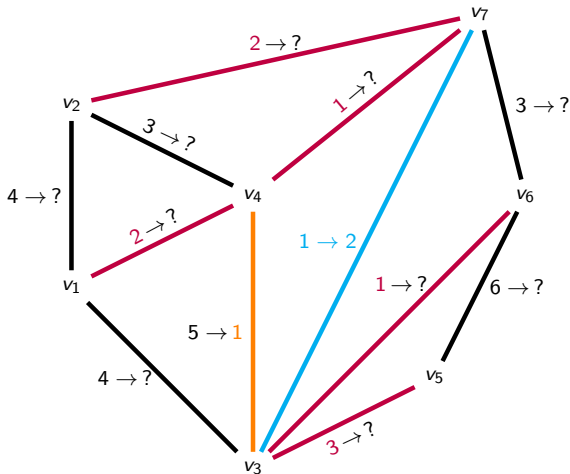
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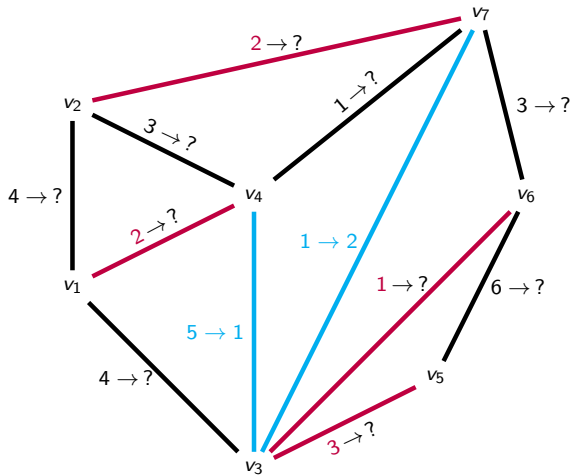
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Theorem

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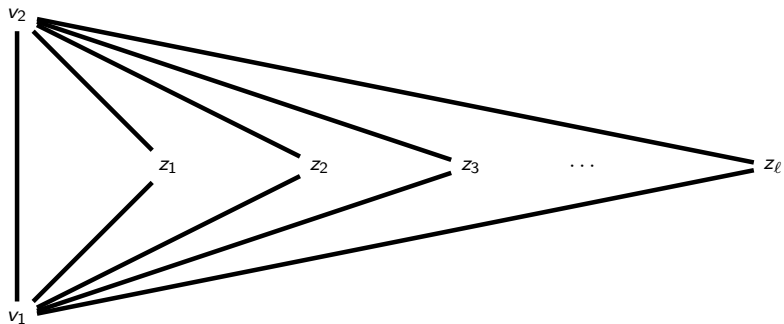
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Theorem

$$CR_{GFTP}(\varepsilon) = 1 + 2\varepsilon$$

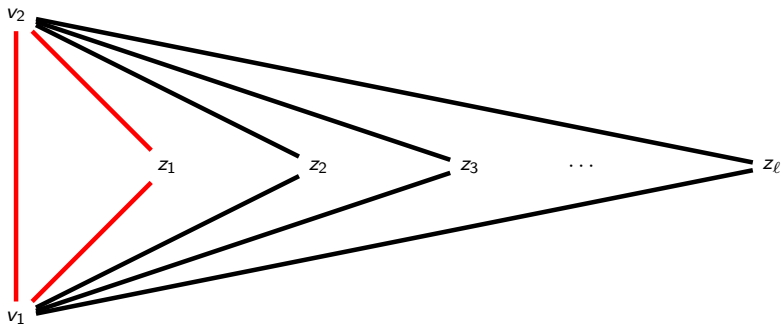
Intuition

Basic graph:

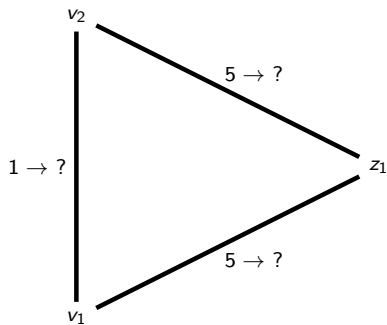


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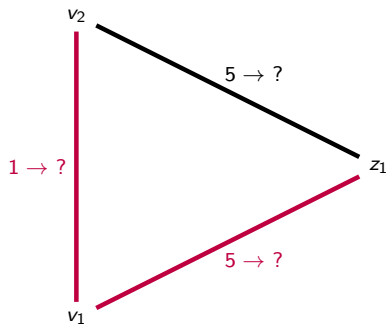
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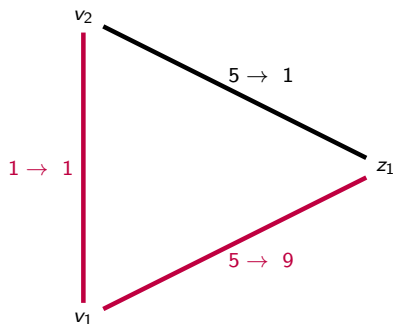
Focussing on a single triangle



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$$\text{FTP}(G) = 9 \cdot \ell + 1$$

$$\text{OPT}(G) = \ell + 1$$

$\eta = 4 \cdot (\ell + 1)$, and so $\varepsilon = 4$.

No algorithm can do better

Theorem

For all $\varepsilon < \frac{1}{2}$, and all online algorithm with predictions ALG , $\text{CR}_{\text{ALG}}(\varepsilon) \geq 1 + 2\varepsilon$.

Random Order Analysis

A weakening of the adversary:

Definition

Let ALG be an online algorithm for a minimization problem Π , and let $I = \langle r_1, r_2, \dots, r_m \rangle$ be an instance of Π . Then, a permutation σ of $\{1, 2, \dots, m\}$ is chosen uniformly at random, and $\sigma(I)$ is presented to ALG . The **random order ratio** of ALG is

$$\text{ROR}_{\text{ALG}} = \inf\{c \mid \exists b : \forall I : \mathbb{E}_{\sigma}[\text{ALG}(\sigma(I))] \leq c \text{OPT}(\sigma(I)) + b\}$$

As competitive ratio, we describe the random order ratio of ALG as a function of ε .

This is the first time random order analysis has been used in online algorithms with predictions.

Separation by Random Order Analysis

This analysis separates FTP and GFTP:

Theorem

$$\text{ROR}_{\text{FTP}}(\varepsilon) = 1 + 2\varepsilon.$$

Theorem

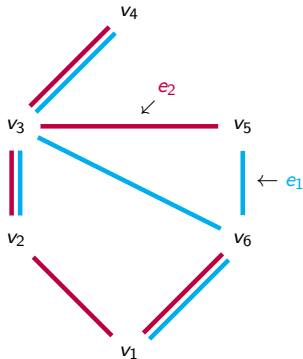
$$\text{ROR}_{\text{GFTP}}(\varepsilon) \leq 1 + (1 + \ln(2))\varepsilon.$$

The idea behind this separation...

Random Order Analysis

Lemma

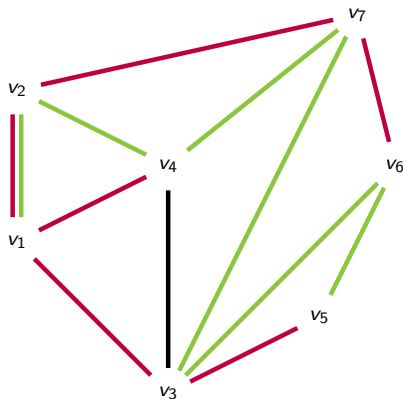
Let G be a graph, and let T_1 and T_2 be two spanning trees of G . Then, for any edge $e_1 \in T_1 \setminus T_2$, there exists an edge $e_2 \in T_2 \setminus T_1$ such that e_2 introduces a cycle into T_1 that contains e_1 , and e_1 introduces a cycle into T_2 that contains e_2 .



How we use this Lemma

Dominate the “random variable” $\text{GFPT}(G, \hat{w}) - \text{OPT}(G)$ online.

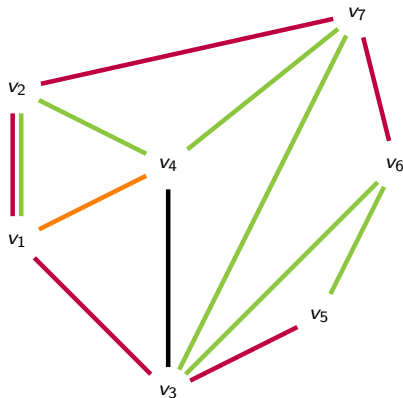
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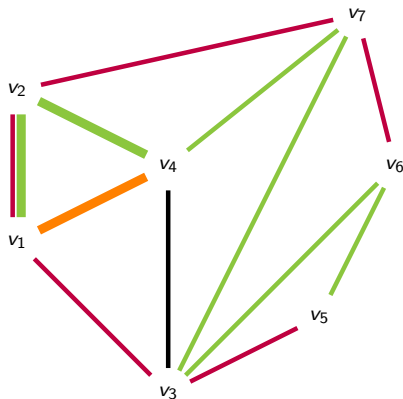
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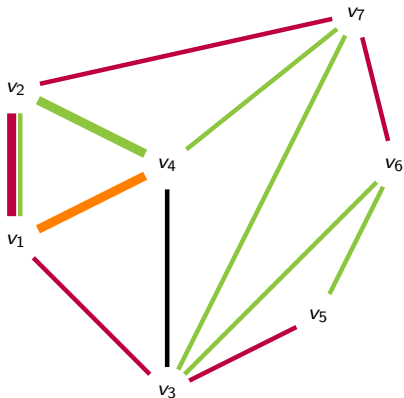
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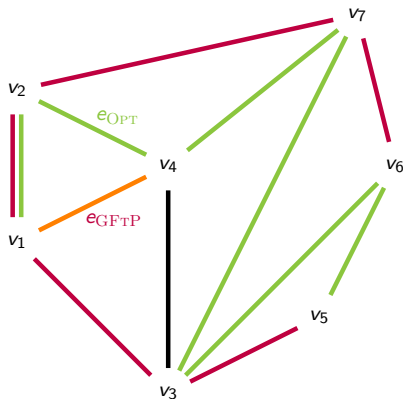
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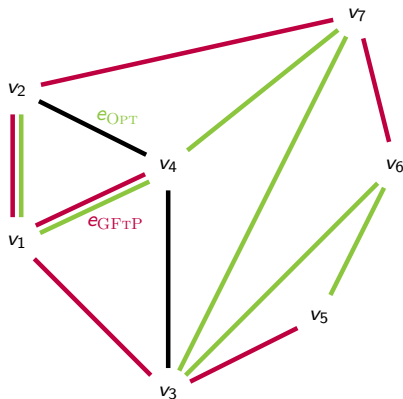
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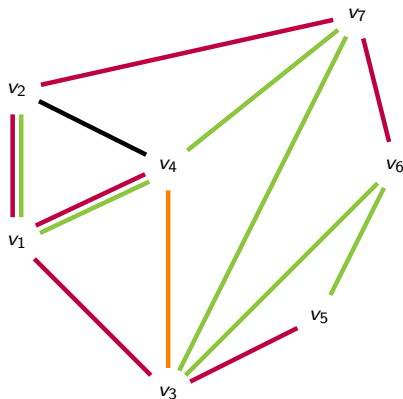


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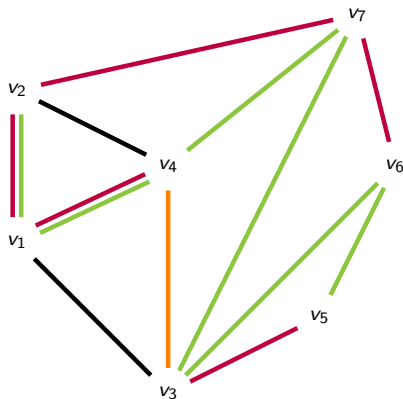


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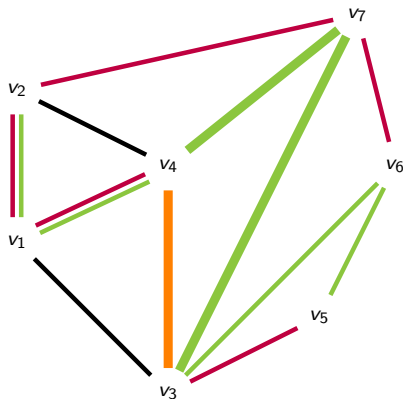


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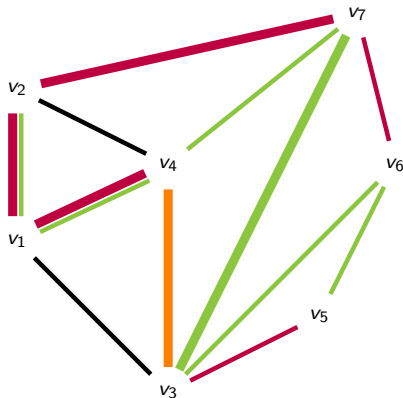


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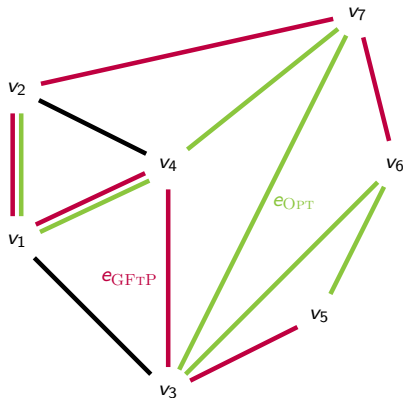


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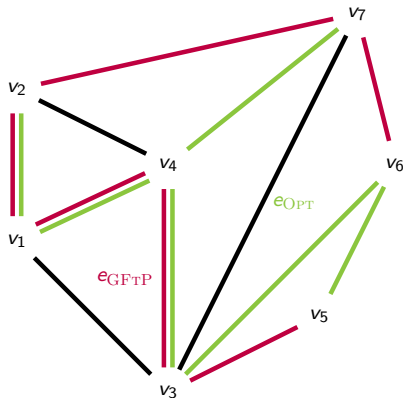


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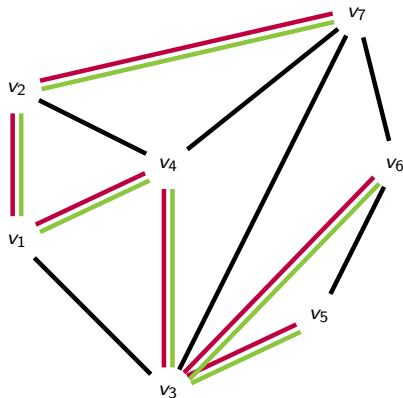


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- $\mathbb{E}[\#\text{dominating edges}] \leq (1 + \ln(2))(n - 1)$, and all distinct.

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Thank you for your attention!